Technical Notes

Minimum Fuel Closed-Loop Translation

D. A. Conrad*
Hughes Aircraft Company, El Segundo, Calif.

Introduction

A CLASS of problems of interest in connection with lunar missions is concerned with translation of a rocket in a vacuum at constant altitude above the lunar surface. For example, it may be required to descend to a point just above the surface, hover for a time sufficient to examine a potential landing site, and then translate if necessary to a more suitable location. From the point of view of surveillance, it may be of benefit to perform such a maneuver at constant altitude, rather than to follow a more efficient ballistic trajectory.

In general, it might be desired to maneuver between prescribed positions and/or velocities in minimum time or, more likely, with as small a fuel expenditure as possible. Leitmann[1] has considered maneuvers between prescribed velocities and has shown that for both minimum fuel and minimum time operation, the thrust is maximum and constant in direction. Sivo et al.,[2] Keller,[6] and Cheng and Conrad[3] have considered minimum fuel rectilinear maneuvers between prescribed positions and zero velocities. In the latter case it was shown for the case of bounded thrust acceleration that the problem reduces to the form of the classical brachistochrone, with the acceleration constraint appearing as a limit on the slope. A closed-form solution was obtained by application of standard techniques of the calculus of variations.

Of considerably greater interest is the problem of closed-loop control. That is, given the current state of the system (in this case position and velocity) and the desired terminal state, what is the optimum control to apply? In this paper such an optimum control law is developed for the case of rectilinear translation to a prescribed terminal position at zero velocity.

The horizontal component of thrust can be developed in two ways. With a single set of engines, the horizontal component is generated by rotating the thrust vector, as shown in Fig. 1. However, simplifications in implementation may result from using a second engine, as shown in Fig. 2. Hence, despite its obvious inefficiency with respect to fuel, this concept has been analyzed previously[2, 6] and will be treated in this paper as well. However, only the analytically more interesting single-engine case will be discussed in detail.†

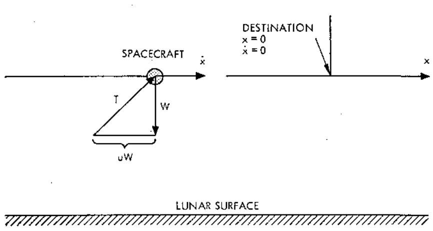

Fig. 1 Geometry: single-engine system.

Presented at the Joint AIAA-IMS-SIAM-ONR Symposium on Control and System Optimization, Monterey, Calif., January 27–29, 1964; revision received December 28, 1964.

* Senior Staff Engineer, Surveyor System Analysis Laboratory, Space Systems Division. Member AIAA.

† The two-engine case is closely related to the minimum-energy attitude control problem considered by Schwartz.[5]

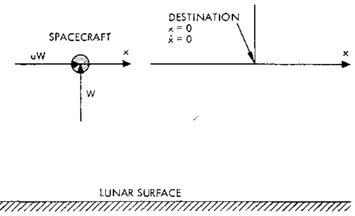

Fig. 2 Geometry: dual-engine system.

Analytical Formulation

We consider the situation shown in Fig. 1, where the horizontal component of thrust is denoted by uW, where W is the weight of the vehicle, and u, the control variable for this problem, is the horizontal component of the thrust acceleration measured in lunar g's. The vehicle is shown in motion toward its destination at the point $x = 0$.

The equation of rectilinear motion is then

$$\ddot{x} = gu \tag{1}$$

where g is the gravitational acceleration. Since the vertical component of thrust acceleration is g and the horizontal component is ug, its magnitude is

$$a_T = g(1 + u^2)^{1/2} \tag{2}$$

from which follows the characteristic velocity

$$\Delta V = \int_{t_0}^{t_f} a_T dt = g \int_{t_0}^{t_f} (1 + u^2)^{1/2} dt \tag{3}$$

where t_0 is to be viewed as the current time and t_f as the terminal time. The thrust acceleration will be assumed to be bounded in magnitude, such that

$$|u| \leq M \tag{4}$$

It is desired to find the control u as a function of the current state (x_0, $\dot{x}_0$) such that at the unprescribed time t_f, the motion reaches the origin ($x = 0$, $\dot{x} = 0$), and the characteristic velocity is as small as possible subject to the constraint (4).

The transformation to the form of the brachistochrone[3] results from treating the distance x as the independent variable and the square of the velocity as the dependent variable as follows. Defining

$$y = \dot{x}^2/2g \tag{5}$$

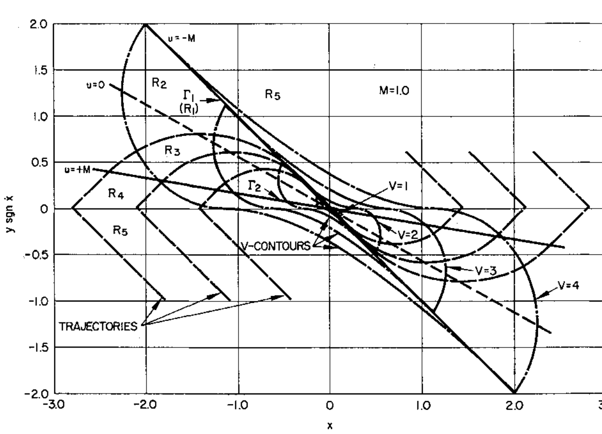

Fig. 3 The phase plane: single-engine case.

then Eqs. (1) and (3) become

$$dy/dx = u \tag{6}$$

$$V(y_0, x_0; u) = \left(\frac{2}{g}\right)^{1/2} \Delta V = \int_{x_0}^{0} \left[\frac{1 + u^2}{y}\right]^{1/2} dx \tag{7}$$

The minimization of (7), subject to (6) and (4), corresponds to the classical brachistochrone problem with a constraint on the magnitude of the slope.

The Optimal Control

The Hamiltonian formulation[4] of optimal control theory was used to obtain the results shown in Fig. 3. The phase plane is divided by two straight lines Γ_1 and Γ_2 into regions of maximum, minimum, and intermediate thrust acceleration as indicated. In the intermediate thrust region the control is given in terms of the current state (x, y) by

$$u + \left[\tan^{-1}u + \tan^{-1}M + \frac{1}{M}\right](1 + u^2) = -\frac{x}{y} \tag{8}$$

when $\dot{x} > 0$. When $\dot{x} < 0$, u is replaced by $-u$ in (8). The control u is continuous, with the transitions between limiting and intermediate thrust accelerations resulting from setting $u = \pm M$ in (8). For $u = -M$, this yields the obvious result

$$y = -Mx \tag{9}$$

for the terminal switching locus Γ_1. The locus Γ_2 of points of transition from maximum to intermediate thrust is obtained from (8) with $u = +M$

$$y = -Mx[2(1 + M^2)(1 + M \tan^{-1}M) - 1]^{-1} \tag{10}$$

From (8) it is seen that loci of constant u are straight lines. When $u = 0$, y is maximum and (8) yields for the locus of maxima

$$y_m = -Mx_m[1 + M \tan^{-1}M]^{-1} \tag{11}$$

In the region of intermediate thrust, the maximum velocity is given in terms of the current velocity and control by

$$y_m = (1 + u^2)y \tag{12}$$

When $u = +M$, $y = y_s$ (the switching point) and

$$y_s = y_m/(1 + M^2) \tag{13}$$

which completes the description of the optimum control.

The resulting motion in the region of intermediate thrust is given by

$$\frac{x_m - x}{y_m} = \pm\left\{\left[\frac{y}{y_m}\left(1 - \frac{y}{y_m}\right)\right]^{1/2} + \cos^{-1}\left(\frac{y}{y_m}\right)^{1/2}\right\} \tag{14}$$

where the positive sign applies to the region where $u > 0$ and the negative where $u < 0$. The values x_m, y_m are obtained from the initial state by means of (8), (12), and (11) with $(x, y) = (x_0, y_0)$. In regions of maximum acceleration, the trajectories are straight lines as indicated in Fig. 3.

The Fuel Requirement

With the optimal control law $u(x, t)$ known, the minimum characteristic velocity is obtained from (7) by straightforward ntegration. The results are most easily expressed in terms of a parameter $\eta = (y/y_m)^{1/2}$ and the constants $\alpha(M) = 1/M + \tan^{-1} M$ and $\beta(M) = (1 + M)^{1/2}/M$

$$
\begin{aligned}
\frac{1}{2y^{1/2}} V(x, y) &= (\beta\eta)/\eta = \beta && \text{in} \quad R_1(\Gamma_1) \\
&= (\alpha - \cos^{-1}\eta)/\eta && \text{in} \quad R_2 \\
&= (\alpha + \cos^{-1}\eta)/\eta && \text{in} \quad R_3 \\
&= (2\alpha - \beta\eta)/\eta && \text{in} \quad R_4 \\
&= (2\alpha + \beta\eta)/\eta && \text{in} \quad R_5
\end{aligned}
\tag{15}
$$

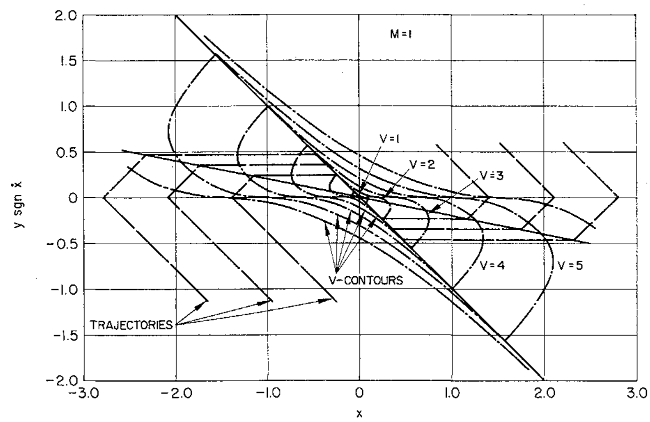

Fig. 4 The phase plane: dual-engine case.

where the regions R_k of the phase plane are indicated in Fig. 3 and the parameter η is given in terms of x and y by

$$
\begin{aligned}
\frac{x}{y} &= -\frac{1}{M} && \text{in} \quad R_1 \\
&= \frac{1}{\eta^2}[\eta(1 - \eta^2)^{1/2} + \cos^{-1}\eta - \alpha] && \text{in} \quad R_2 \\
&= \frac{1}{\eta^2}[-\eta(1 - \eta^2)^{1/2} - \cos^{-1}\eta - \alpha] && \text{in} \quad R_3 \\
&= \frac{1}{\eta^2}\left(\frac{1}{M}\eta^2 - 2\alpha\right) && \text{in} \quad R_4 \\
&= \frac{1}{\eta^2}\left(\frac{1}{M}\eta^2 - 2\alpha\right) && \text{in} \quad R_5
\end{aligned}
\tag{16}
$$

Based on (15) and (16), contours of constant fuel consumption were computed and are shown in Fig. 3.

By following Kalman's approach to the Bushaw problem,[4] it is possible to demonstrate that the fuel requirement given by (15) is the true minimum.

Translation with Two Engines

For the case of the two-engine system shown in Fig. 2, the results are of similar form and are shown in Fig. 4. The only change in the problem formulation is that $(1 + u^2)^{1/2}$ in (2), (3), and (7) is replaced by $(1 + |u|)$.

As in the single-engine case, the phase plane is divided by two straight lines Γ_1 and Γ_2 into regions of maximum, minimum, and intermediate thrust acceleration. However, in this case the intermediate level of control is zero, that is, the vehicle coasts, sustained by the vertically directed engine, with the horizontal engine off.

The equations of the lines Γ_1 and Γ_2 are, respectively,

$$y = -Mx \tag{17}$$

$$y = -Mx/(4M + 1) \tag{18}$$

The minimum characteristic velocity is given directly in terms of current position and velocity by

$$
\begin{aligned}
\frac{V}{2y^{1/2}} &= 1 + \frac{1}{M} && \text{in} \quad R_1 \\
&= -\frac{x}{2y} + 1 + \frac{1}{2M} && \text{in} \quad R_2 \\
&= \frac{1}{M}\left[2(1 + 2M)\left(1 - M\frac{x}{y}\right)\right]^{1/2} - \left(1 + \frac{1}{M}\right) && \text{in} \quad R_3 \text{ and } R_4 \\
&= \frac{1}{M}\left[2(1 + 2M)\left(1 - M\frac{x}{y}\right)\right]^{1/2} + \left(1 + \frac{1}{M}\right) && \text{in} \quad R_5
\end{aligned}
\tag{19}
$$

where, as before, $V = (2/g)^{1/2}\Delta V$.

In Fig. 4, trajectories and constant fuel contours are shown for comparison with the single-engine results of Fig. 3. It is apparent that substantial fuel penalties are associated with the two-engine concept.

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A New Concept in Rocket Engine Baffles

W. A. Sirignano*
Princeton University, Princeton, N. J.

AND

W. C. Strahle†
Aerospace Corporation, San Bernardino, Calif.

Nomenclature

$R(x)$	=	profile of baffle
t	=	time
x,y	=	transverse dimensions
$\epsilon Y(x)$	=	width of cavity
ϵ	=	maximum width of cavity
η	=	translated y dimension
φ	=	velocity potential
Φ	=	amplitude of velocity potential
ω	=	frequency

Introduction

It is well known that the occurrence of transverse-spinning mode, high-frequency combustion instability in the liquid propellant rocket engine can often be eliminated by the use of baffles. These baffles are solid surfaces, which protrude from the injector plate, are perpendicular to it, and extend some determined distance into the rocket combustion chamber. The pattern or arrangement of these baffles required to prevent combustion instability cannot as yet be predicted a priori, and generally it may be said that such design is still an art rather than a science.

In some situations, however, there are certain rational guidelines to this design. It appears for some configurations that a design that segments the chamber into smaller sections will be successful in preventing the spinning mode in the chamber although the standing mode may still be present.[1] This is true even if the protrusion distance is rather small as compared to the chamber length provided that the baffle protrudes part of the way into a region of active combustion.

Received July 1, 1964; revision received December 17, 1964.

* Research Staff Member, Department of Aerospace and Mechanical Sciences. Associate Member AIAA.

† Member of Technical Staff, Propulsion Department; formerly Research Associate, Department of Aerospace and Mechanical Sciences, Princeton University, Princeton, N. J. Associate Member AIAA.

If this is true, then segmenting a chamber by baffles changes the acoustic modes of oscillation of the full chamber to those permitted by the baffle cavities. Even standing modes can be altered in type, and in frequency of oscillation, from those permitted for the full chamber. This is important as larger and larger rocket engines are considered since, as a rocket engine size increases, the natural frequencies of the chamber decrease. However, it is well known that the higher the frequency, the more stable will be the engine.[2, 3] This is caused by the frequency-dependent energy feedback to an acoustic wave from the combustion process. It appears that above a certain frequency (dependent upon propellant and injector type) this feedback is insufficient to sustain the wave by overcoming the damping caused by the exit nozzle. Therefore, if baffles really do alter the acoustic nature of the chamber in the manner described previously (note that there are some observed exceptions to this postulate), it is clear that a sufficiently large number of cavities can increase the frequencies of the allowable modes to a point where combustion instability is not possible in the transverse modes.

If this design principle is adopted, it is reasonable to ask if there is an optimum baffle pattern or shape. Of course, "optimum" must be defined. By this it will be meant that 1) the natural fundamental frequency of the cavities is as high as possible for any given number of cavities; 2) the baffles must not have a complicated shape that would entail manufacturing problems or cause excessive engine weight; and 3) there will be a dispersive device in the baffle shape so that, given an oscillation at the natural frequency of the cavity, the amplitude will be less than that at the point of maximum amplitude. It will be shown that, given the previously mentioned assumption concerning the effect of a baffle, a shape meeting the specifications does exist.

Analysis

The effect of the combustion process in determining the natural frequency of both longitudinal and transverse oscillations is of the order of the mean Mach number of the flow and is considered negligible as compared to unity. This would be a small correction even for large Mach numbers.[1, 3] For transverse oscillations, variations in the longitudinal direction also are of the same order and are neglected here.[3] A linearized analysis gives a satisfactory first approximation to the natural frequency of oscillation within the cavity. The oscillation is described by the wave equation, and, if φ is the velocity potential, t is time, nondimensionalized by chamber diameter divided by speed of sound, and x and y are the transverse dimensions in a Cartesian system, nondimensionalized by chamber diameter, a solution assumed to be of the form

$$\varphi = \Phi(x, y)e^{i\omega t}$$

will be described by the Helmholtz equation

$$(\partial^2\Phi/\partial x^2) + (\partial^2\Phi/\partial y^2) + \omega^2\Phi = 0 \qquad (1)$$

Now the frequency ω would be determined by the shape of the cavity by means of the application of boundary conditions to

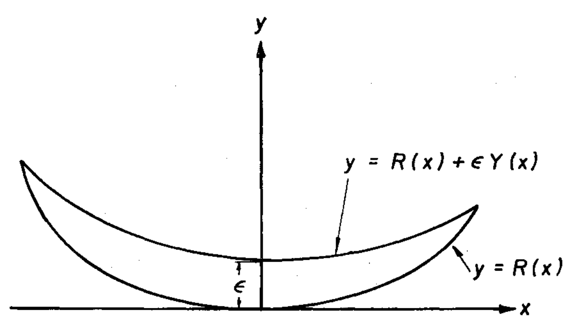

Fig. 1 The general geometrical configuration.